



9 Linear operators on finite-dimensional vector spaces





Introduction

- In this topic, we will
 - Review some of the properties seen in the last chapter
 - Now focus on linear operators on finite-dimensional vector spaces
 - Describe the properties we will examine in this chapter





Linear operators on $\mathbf{R}^n$

- Suppose that $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a linear operator
 - That is, $A \in \mathcal{L}(\mathbf{R}^n)$
- As we have seen:
 - All such linear operators can be represented by square matrices
 - All square matrices represent linear operators





Properties of linear maps

- Given a linear map, we have already seen that we can describe
 - The range
 - The null space
- From these, we may determine whether a linear map is
 - Onto
 - One-to-one
- We deduced that
 - If the dimension of the range equals the dimension of the co-domain, the map is onto
 - If the dimension of the null space is zero, the map is one-to-one





Properties of linear maps

- For example, if $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$ and
 - If $n > m$,
 - There are at least $n - m > 0$ free variables
 - The null space has dimension at least $n - m > 0$
 - The linear map is not one-to-one
 - If $n < m$,
 - The maximum number of basis vectors of the range is $n < m$,
 - The dimension of the range is less than m
 - The linear map is not onto





Properties of linear operators

- Thus, only a linear operator $A: \mathbf{R}^n \rightarrow \mathbf{R}^n$ on finite-dimensional vector spaces may possibly be both one-to-one and onto
 - If $\text{rank}(A) = n$, then it is one-to-one and onto
 - If $\text{rank}(A) < n$, then it is neither one-to-one nor onto
 - These are an important classification of linear maps
- Given a square matrix,
 - we will find three additional properties of the matrix:
 - The determinant
 - The trace
 - The inverse if it exists





Summary

- Following this topic, you now
 - Understand what will be covered in the next chapter
 - Know we are looking at properties of square matrices
 - Are aware that these properties will allow us to describe the behavior of these matrices





References

- [1] https://en.wikipedia.org/wiki/Linear_map





Acknowledgments

None so far.





Colophon

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